Minimización de costos Cobb Douglas

Diego López Tamayo [*]

Contents

Procedimiento

$$Q(K,L) = AL^{\alpha}K^{\beta}$$

$$CT = p_L \cdot L + p_K \cdot K$$

Antes de plantear el problema de optimización, notar que podemos obtener la productividad marginal del trabajo L y capital K y reescribirlas de las siguientes formas útiles:

$$\frac{\partial Q}{\partial L} = A\alpha L^{\alpha-1}K^{\beta} = \alpha AL^{\alpha}L^{-1}K^{\beta} = \frac{\alpha AL^{\alpha}K^{\beta}}{L} = \alpha\frac{Q}{L}$$

$$\frac{\partial Q}{\partial K} = A\beta L^{\alpha}K^{\beta-1} = \beta AL^{\alpha}K^{\beta}K^{-1} = \frac{\beta AL^{\alpha}K^{\beta}}{K} = \beta\frac{Q}{K}$$

Notar que si despejamos para α y β obtenemos las elasticidades insumo de la producción: $\alpha = \frac{\partial Q}{\partial L}\frac{L}{Y}$ y $\beta = \frac{\partial Q}{\partial K}\frac{K}{Y}$ Si $\alpha = 2$, entonces un aumento del 10% en la mano de obra (L) dará como resultado un aumento del 20% en la producción (Y).

Planteamos ahora el **problema de minimización restringida**:

Resolución por método de Multiplicadores de Lagrange

$$\underset{K,L}{\text{Min}}\ CT = p_L \cdot L + p_K \cdot K$$

$$\text{sujeto a } Q = AL^{\alpha}K^{\beta}$$

$$\mathbf{L} = p_L \cdot L + p_K \cdot K + \lambda[Q - AL^{\alpha}K^{\beta}]$$

Obtenemos las condiciones de primer orden

$$\frac{\partial \mathbf{L}}{\partial L} = 0 \Rightarrow p_L = \lambda\alpha AL^{\alpha-1}K^{\beta}$$

[*]El Colegio de México, diego.lopez@colmex.mx

$$\frac{\partial \mathbf{L}}{\partial K} = 0 \Rightarrow p_K = \lambda \beta A L^{\alpha} K^{\beta - 1}$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 0 \Rightarrow Q = L^{\alpha} K^{\beta}$$

Dividimos las primeras dos CPO para simplificar el problema:

$$\frac{\frac{\partial \mathbf{L}}{\partial K}}{\frac{\partial \mathbf{L}}{\partial L}} = \frac{p_K}{p_L} = \frac{\lambda \beta A L^{\alpha} K^{\beta-1}}{\lambda \alpha A L^{\alpha-1} K^{\beta}}$$

Simplificamos y resolvemos para cualquiera de los dos insumos. (Aquí resolvemos para L)

$$\frac{p_K}{p_L} = \frac{\beta}{\alpha} \frac{L}{K}$$

$$L = \frac{p_K}{p_L} \frac{\alpha}{\beta} K$$

Sustituímos este valor en nuestra tercer CPO (que es nuestra función de producción)

$$Q - A L^{\alpha} K^{\beta} = 0$$

$$Q - A \left(\frac{p_K}{p_L} \frac{\alpha}{\beta} K \right)^{\alpha} K^{\beta} = 0$$

$$Q = A \left(\frac{p_K}{p_L} \frac{\alpha}{\beta} \right)^{\alpha} K^{\alpha+\beta}$$

$$\frac{Q}{A} \left(\frac{p_K}{p_L} \frac{\alpha}{\beta} \right)^{-\alpha} = K^{\alpha+\beta}$$

$$K = \left(\frac{Q}{A} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_K}{p_L} \frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha+\beta}}$$

$$K^{*} = \left(\frac{Q}{A} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_L}{p_K} \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}}$$

Ahora podemos resolver para $L = \frac{p_K}{p_L} \frac{\alpha}{\beta} K$

$$L = \frac{p_K}{p_L} \frac{\alpha}{\beta} \left(\frac{Q}{A} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_L}{p_K} \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}}$$

$$L = \frac{p_K}{p_L} \frac{\alpha}{\beta} \left(\frac{Q}{A} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_L}{p_K} \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}}$$

$$L = \left(\frac{Q}{A} \right)^{\frac{1}{\alpha+\beta}} \frac{p_K}{p_L} \left(\frac{p_L}{p_K} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{\alpha}{\beta} \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}}$$

Notar que podemos expresar $\frac{p_K}{p_L}$ como $\left(\frac{p_L}{p_K} \right)^{-1} = \left(\frac{p_L}{p_K} \right)^{\frac{-\alpha+\beta}{\alpha+\beta}}$

$$L = \left(\frac{Q}{A} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_K}{p_L} \right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha+\beta}}$$

$$L^* = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_K}{p_L}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$$

Con los resultados en parámetros de $K^* = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_L}{p_K}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$ y $L^* = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_K}{p_L}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$ podemos obtener la **función de costos mínimos** o función valor del problema de optimización:

$$C_{min} = p_K \cdot K^* + p_L \cdot L^*$$
$$C_{min} = p_K \cdot \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_L}{p_K}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} + p_L \cdot \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_K}{p_L}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$$

Factorizamos $\left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}}$

$$C_{min} = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(p_K \left(\frac{p_L}{p_K}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} + p_L \left(\frac{p_K}{p_L}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\right)$$

Simplificamos los precios de los factores y factorizamos

$$C_{min} = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} p_K^{\frac{\beta}{\alpha+\beta}} p_L^{\frac{\alpha}{\alpha+\beta}} \left(\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\right)$$

Caso particular con valores arbitrarios

Utilizando los resultados para nuestro problema $\alpha = \frac{3}{4}$ $\beta = \frac{1}{4}$ $Q = 800$ $p_K = 4$ $p_L = 2$

$$K^* = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_L}{p_K}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} = \left(\frac{800}{1}\right)^{\frac{1}{(3/4)+(1/4)}} \left(\frac{2}{4}\frac{(1/4)}{(3/4)}\right)^{\frac{(3/4)}{(3/4)+(1/4)}}$$

$$L^* = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{p_K}{p_L}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} = \left(\frac{800}{1}\right)^{\frac{1}{(3/4)+(1/4)}} \left(\frac{4}{2}\frac{(3/4)}{(1/4)}\right)^{\frac{(1/4)}{(3/4)+(1/4)}}$$

$$C_{min} = \left(\frac{800}{1}\right)^{\frac{1}{(3/4)+(1/4)}} 4^{\frac{(1/4)}{(3/4)+(1/4)}} 2^{\frac{(3/4)}{(3/4)+(1/4)}} \left(\left(\frac{(1/4)}{(3/4)}\right)^{\frac{(3/4)}{(3/4)+(1/4)}} + \left(\frac{(3/4)}{(1/4)}\right)^{\frac{(1/4)}{(3/4)+(1/4)}}\right)$$

Cálculo mediante paquetería R

```
#Parámetros a utilizar
p_K=4
p_L=2
Q=800
A=1
alpha=(3/4)
beta=(1/4)

#Cálculo de resultados de minimización de costos
K_min <- (Q/A)^(1/(alpha+beta))*((p_L/p_K)*(beta/alpha))^(alpha/(alpha+beta))
L_min <- (Q/A)^(1/(alpha+beta))*((p_K/p_L)*(alpha/beta))^(beta/(alpha+beta))
Costo_min <- ((Q/A)^(1/(alpha+beta))*(p_K)^(beta/(alpha+beta)) *
                (p_L)^(alpha/(alpha+beta)))*((beta/alpha)^(alpha/(alpha+beta)) +
                                              (alpha/beta)^(beta/(alpha+beta)))
```

```
##                           [,1]
## Capital (K) óptimo  208.6779
## Trabajo (L) óptimo 1252.0677
## Costo mínimo       3338.8471
```